

WKB for a damped spin

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Abstract. The master equation for a damped spin well known from the theory of superradiance, is written as a finite-difference equation and solved by a WKB-like method. The propagator thus obtained looks like the van Vleck propagator of a certain classical Hamiltonian system with one degree of freedom. A new interpretation is provided of the temporal broadening of initially sharp probability distributions as the analogue of the spreading of the quantum mechanical wave packet.

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1 Introduction

This is the second of a series of papers concerned with dissipative motion of a large spin which may be realized with many identical collectively radiating two-level atoms. The large spin (the “Bloch vector” of quantum optics) has conserved length such that its classical state can be described by two angles θ and φ . The classical dynamics is that of an overdamped pendulum,

$$\begin{aligned} \dot{\theta} &= \sin \theta, \\ \tan \frac{\theta(\tau)}{2} &= e^\tau \tan \frac{\theta(0)}{2}, \end{aligned} \quad (1.1)$$

$$\varphi(\tau) = \text{const.}$$

In geodesic jargon we could think of θ and φ as defining latitude and longitude and speak of creeping motion towards the south pole along a great circle.

The starting point of the first paper [1], which we shall refer to as I, was the exact solution of the “superradiance master equation” (see below) in the form of a Laplace integral [2]. We evaluated that integral in a saddle-point approximation and derived uniform asymptotics of the dissipative propagator.

In the present paper we employ a different strategy far less dependent on the specific properties of the problem. We observe that in the limit of a large number N of atoms the master equation becomes a finite-difference equation with a small step, amenable to solution by an approximation of the WKB type. The propagator solution thus obtained takes the form of a van Vleck propagator involving

the action of a certain classical Hamiltonian system with one degree of freedom. We find the pertinent Hamilton equations to be equivalent to the saddle-point equation of I. The WKB approximation entails a new interpretation of the temporal broadening of initially sharp probability distributions as the analogue of the spreading of a quantum mechanical wave packet.

As in I we shall employ the basis formed by the eigenstates $|jm\rangle$ of \mathbf{J}^2 and J_z with the respective eigenvalues $j(j+1)$ and m . The quantum number j can take on integer or half integer positive values (up to half the number N of two-level atoms) and for fixed j the quantum number m runs in unit steps from $-j$ to $+j$. We are interested in the limit of a large number of atoms or, more specifically, of a large spin,

$$\sqrt{j(j+1)} \approx j + 1/2 \equiv J \gg 1. \quad (1.2)$$

Denoting the density matrix elements by $\langle j, m+k | \rho(\tau) | j, m-k \rangle = \rho_m^k(\tau)$ we can write the master equation under study as

$$J \frac{d\rho_m^k}{d\tau} = \sqrt{g_{m+k+1}g_{m-k+1}} \rho_{m+1}^k - (g_m - k^2) \rho_m^k, \quad (1.3)$$

where τ is a suitably scaled dimensionless time and g_m denotes the “rate function”

$$g_m = j(j+1) - m(m-1). \quad (1.4)$$

The particular solution $\rho_m^k(\tau)$ satisfying the initial condition $\rho_m^k(\tau=0) = \delta_{mn}$ with a certain n is called the dissipative propagator and denoted as $D_{mn}^k(\tau)$. The solution for an arbitrary initial density matrix then is $\rho_m^k(\tau) = \sum_{n=-j}^j D_{mn}^k(\tau) \rho_n^k(0)$.

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As shown in I, the master equation does not couple density matrix elements with different skewness k . In particular, the probabilities ($k = 0$) can be solved for independently of the coherences ($k \neq 0$). The elements of the “coherence propagator” D_{mn}^k with $k \neq 0$ are connected by an elementary relation with those of the density propagator D_{mn}^0 (see below, Eq. (A.1)). When dealing with the probabilities ρ_m^0 and their propagator D_{mn}^0 we shall drop the superscript $k = 0$.

2 Semiclassical asymptotics

2.1 Finite-difference equation

In the limit of large J it is convenient to introduce a new independent variable μ and its increment Δ as

$$\mu = m/J, \quad \Delta = J^{-1}. \quad (2.1)$$

In the classical limit μ would become continuous in the range $-1 \dots 1$. For our semiclassical perspective μ remains discrete but neighboring values are separated by Δ . The master equation (1.3) for the densities (case $k = 0$) becomes a finite difference equation,

$$\frac{\partial \rho(\mu, \tau)}{\partial \tau} = J \left[\left(1 - \mu^2 - \mu \Delta - \frac{\Delta^2}{4} \right) \rho(\mu + \Delta, \tau) - \left(1 - \mu^2 + \mu \Delta - \frac{\Delta^2}{4} \right) \rho(\mu, \tau) \right]. \quad (2.2)$$

2.2 WKB ansatz

The WKB formalism for finite-difference equations with a small step is well established. The general theory has been worked out mostly by Maslov [3] whose notations we shall adhere to. The WKB method for ordinary second-order difference equations was extensively used to study the eigenvalues of huge tridiagonal matrices occurring in the theory of Rydberg atoms in external fields [4]. Closer to our topic, the leading (exponential) term in the semiclassical solution of master equations of the type (2.2) was obtained in [5]. We follow the same lines but go a step further by also establishing the preexponent, as is indeed necessary to get meaningful results for most quantities of interest.

Let us look for a solution of (2.2) in a form reminiscent of the WKB wave function in a classically forbidden domain,

$$\rho(\mu, \tau) = A(\mu, \tau) e^{JS(\mu, \tau)}. \quad (2.3)$$

Here the prefactor A and the “action” S are smooth functions satisfying the initial conditions

$$S(\mu, 0) = S_0(\mu), \quad A(\mu, 0) = A_0(\mu). \quad (2.4)$$

In our case the absence of the imaginary unit from the exponential does not signal the sojourn of our spin in forbidden terrain but simply accounts for the dissipative character of the dynamics in consideration. Incidentally, due to the presence of the large parameter J even modest changes of S_0 are reflected in wild fluctuations of $\rho(\mu, 0)$; the ansatz therefore does not limit our discussion to smooth probability distributions.

No loss of generality is incurred by assuming the function $S(\mu, \tau)$ independent of J since the prefactor $A(\mu, \tau)$ may pick up all dependence on J . We represent the latter by an expansion in powers of $\Delta = J^{-1}$

$$A(\mu, \tau) = A^{(0)}(\mu, \tau) + A^{(1)}(\mu, \tau) \Delta + A^{(2)} \Delta^2 + \dots \quad (2.5)$$

The master equation (2.2) then allows to determine $S, A^{(0)} \dots$ recursively. We shall need the equations for the action and the zero-order prefactor,

$$\frac{\partial S}{\partial \tau} + (1 - \mu^2) \left[1 - e^{\frac{\partial S}{\partial \mu}} \right] = 0, \quad (2.6)$$

$$\begin{aligned} \left(\frac{\partial}{\partial \tau} - e^{\frac{\partial S}{\partial \mu}} (1 - \mu^2) \frac{\partial}{\partial \mu} \right) \ln A^{(0)} \\ = e^{\frac{\partial S}{\partial \mu}} \left[(1 - \mu^2) \frac{1}{2} \frac{\partial^2 S}{\partial \mu^2} - \mu \right] - \mu. \end{aligned} \quad (2.7)$$

We shall neglect all higher-order corrections to the zero-order prefactor.

2.3 Hamiltonian dynamics

We may consider (2.6) as the Hamilton-Jacobi equation for a classical system with one degree of freedom and the Hamiltonian function

$$H(\mu, p) = (1 - \mu^2)(1 - e^p). \quad (2.8)$$

The canonical equations of motion

$$\dot{\mu} = \frac{\partial H}{\partial p} = -(1 - \mu^2)e^p, \quad \dot{p} = -\frac{\partial H}{\partial \mu} = 2\mu(1 - e^p)$$

are easily integrated. In the resulting “Hamiltonian” trajectories,

$$\tau = \frac{1}{2a} \ln \frac{(a + \nu)(a - \mu)}{(a - \nu)(a + \mu)}, \quad (2.9)$$

$$p = \ln \frac{a^2 - \mu^2}{1 - \mu^2}, \quad (2.10)$$

we denote by ν the initial coordinate; the name “Hamiltonian” is meant to distinguish these solutions from the trajectories of the overdamped pendulum (see below). The second integration constant, a , determines the “energy” $E = H(\mu, p)$ as

$$a \equiv \sqrt{1 - E}. \quad (2.11)$$

Rather remarkably, the Hamiltonian trajectory (2.9) coincides with the saddle-point equation incurred in I when examining the asymptotics of the Laplace representation of the propagator. The saddle-point parameter a reappears in a new “energetic” role. For later reference we note the nonnegative “speed”

$$\dot{\mu} = -(a^2 - \mu^2). \quad (2.12)$$

A special class of trajectories has zero initial momentum, $p(\tau = 0) = 0$, therefore vanishing energy E and $a = 1$. Their Hamiltonian trajectories,

$$\tau = \frac{1}{2} \ln \frac{(1 + \nu)(1 - \mu)}{(1 - \nu)(1 + \mu)}, \quad p(t) = 0, \quad (2.13)$$

are in fact just those of the classical overdamped pendulum (1.1), disguised by $\mu = \cos \theta$. They involve the canonical momentum as conserved with the value zero.

The semiclassical quantum effects which our Hamiltonian dynamics imparts to the spin through the WKB ansatz (2.3) may be seen in the existence of the Hamiltonian trajectories (2.9) not included in the special class (2.13).

2.4 Solution of the Hamilton-Jacobi equation

The familiar relation between canonical momentum and action,

$$p = \frac{\partial S(\mu, \tau)}{\partial \mu}, \quad (2.14)$$

implies $p_0(\nu) = \partial S_0(\nu) / \partial \nu$ at the initial moment $\tau = 0$. Since S_0 is fixed by the initial density distribution this latter equation uniquely associates an initial momentum with the initial coordinate ν . One and only one Hamiltonian trajectory $\mu(\tau; \nu)$ thus passes at $\tau = 0$ through the initial coordinate ν , provided of course that we consider the initial probabilities as imposed. Conversely, we can find the initial coordinate $\nu = \nu(\mu, \tau)$ from which the current coordinate μ is reached at time τ along the unique Hamiltonian trajectory.

The action $S(\mu, \tau)$ can now be obtained by integration along the trajectory just discussed,

$$S(\mu, \tau) = \left[S_0(\nu) + \int_{\nu}^{\mu} p d\mu - E\tau \right]_{\nu=\nu(\mu, \tau)}. \quad (2.15)$$

We use the explicit form of the Hamiltonian trajectories (2.10) to do the integral. For the sake of later convenience we express the resulting action in terms of the auxiliary functions

$$\begin{aligned} \sigma(a, \mu, \nu) &= (\nu + a) \ln(\nu + a) - (\mu + a) \ln(\mu + a) \\ &\quad - (a - \nu) \ln(a - \nu) + (a - \mu) \ln(a - \mu), \end{aligned} \quad (2.16)$$

$$\Phi(\mu, \nu, \tau) = [\sigma(1, \mu, \nu) - \sigma(a, \mu, \nu) + \tau(a^2 - 1)]_{a=a(\mu, \nu, \tau)} \quad (2.17)$$

as

$$S(\mu, \tau) = [S_0(\nu) + \Phi(\mu, \nu, \tau)]_{\nu=\nu(\mu, \tau)}. \quad (2.18)$$

In the definition (2.17) of the function Φ the parameter a must, as indicated in the notation above, be read as a function of the initial and final values of the coordinate since these are at present considered as defining a Hamiltonian trajectory. We may interpret the function Φ as the action accumulated along the Hamiltonian trajectory in question. Its derivatives with respect to μ and ν accordingly give the final and initial momenta,

$$\frac{\partial \Phi}{\partial \mu} = \ln \frac{a^2 - \mu^2}{1 - \mu^2} = p, \quad (2.19)$$

$$\frac{\partial \Phi}{\partial \nu} = -\ln \frac{a^2 - \nu^2}{1 - \nu^2} = -p_0. \quad (2.20)$$

The Hamiltonian trajectory $\mu(\tau; \nu)$ can be regarded as the solution with respect to μ of the equation

$$\frac{\partial \Phi(\mu, \nu, \tau)}{\partial \nu} = -\frac{\partial S_0(\nu)}{\partial \nu}. \quad (2.21)$$

2.5 WKB prefactor

The expression (2.7) for the prefactor can be simplified using the notion of the full time derivative of a function $f(\mu, \tau)$ along the Hamiltonian trajectory $\mu(\tau; \nu)$,

$$\frac{df(\mu, \tau)}{d\tau} = \frac{\partial f(\mu(\tau, \nu), \tau)}{\partial \tau} \Big|_{\nu} = \frac{\partial f}{\partial \tau} \Big|_{\mu} + \dot{\mu} \frac{\partial f}{\partial \mu} \Big|_{\tau},$$

since the left hand side in (2.7) is just the full time derivative dA/dt (see Eq. (2.14)). We next introduce the Jacobian

$$Y(\tau; \nu) = \frac{\partial \mu(\tau, \nu)}{\partial \nu}, \quad Y(0; \nu) = 1, \quad (2.22)$$

and a new exponent $B(\mu, \tau)$ to rewrite the prefactor as

$$A = \frac{e^{B(\mu, \tau)}}{\sqrt{Y}}. \quad (2.23)$$

The full time derivative of Y can be transformed to

$$\begin{aligned} \frac{dY}{d\tau} &= \frac{\partial^2 \mu}{\partial \tau \partial \nu} = \frac{\partial}{\partial \nu} \dot{\mu} = \frac{\partial}{\partial \nu} \frac{\partial H(\mu, p)}{\partial p} \\ &= \frac{\partial \mu}{\partial \nu} \left(\frac{\partial^2 H}{\partial \mu \partial p} + \frac{\partial p}{\partial \mu} \frac{\partial^2 H}{\partial p^2} \right) \\ &= Y \left[\frac{\partial^2 H}{\partial \mu \partial p} + \frac{\partial^2 S(\mu, \tau)}{\partial \mu^2} \frac{\partial^2 H}{\partial p^2} \right] \\ &= Y \exp \left(\frac{\partial S}{\partial \mu} \right) \left[2\mu - (1 - \mu^2) \frac{\partial^2 S}{\partial \mu^2} \right]. \end{aligned} \quad (2.24)$$

So equipped we find the simple evolution equation $\frac{dB}{d\tau} = -\mu$ for the function $B(\mu, \tau)$ which can be integrated along

the trajectory to give

$$\begin{aligned} B(\mu, \tau) &= - \int_0^\tau \mu d\tau + \ln A(\nu, 0) \\ &= - \frac{1}{2} \ln \frac{a^2 - \mu^2}{a^2 - \nu^2} + \ln A(\nu, 0). \end{aligned} \quad (2.25)$$

We thus arrive at the asymptotic solution of the Cauchy problem for our master equation with the initial condition (2.4),

$$\rho(\mu, \tau) = \frac{1}{\sqrt{\frac{\partial \mu(\tau, \nu)}{\partial \nu}}} \sqrt{\frac{a^2 - \nu^2}{a^2 - \mu^2}} e^{J\Phi(\mu, \nu, \tau)} \rho(\nu, 0), \quad (2.26)$$

where ν, a are meant as functions of μ and τ as explained above.

2.6 Narrow versus broad initial distributions

We have in effect constructed the solution (2.26) by the method of characteristics. At any rate, the density at a certain point is obtained from the initial density by transport along the Hamiltonian trajectory and acquires a factor consisting of an exponential and a prefactor.

Let us show that the exponential factor is smaller than or at most equal to unity. According to (2.19) the extremum of Φ regarded as a function of the final coordinate μ with ν, τ fixed occurs when $a = 1$, *i.e.* when μ moves like for the overdamped pendulum, $\mu(\tau; \nu, a = 1) \equiv \mu_{pend}(\tau; \nu)$ as defined by (2.13) or (1.1). At the extremum we have $\Phi = 0$ and

$$\left. \frac{\partial^2 \Phi(\mu, \nu, \tau)}{\partial \mu^2} \right|_{\mu=\mu_{pend}} = - \left. \frac{\Xi}{(1 - \mu^2)} \right|_{\mu=\mu_{pend}} \quad (2.27)$$

where Ξ is a positive function defined as

$$\Xi(\mu, \nu, \tau) = \frac{1}{2a^2} \left(\tau + \frac{\nu}{a^2 - \nu^2} - \frac{\mu}{a^2 - \mu^2} \right) \Big|_{a=a(\mu, \nu, \tau)}. \quad (2.28)$$

But since Φ has a negative second derivative its extremum is indeed a maximum, hence $\Phi \leq 0$. We should mention that by using (2.27) it is easy to calculate the integral of the density (2.26) over μ by the saddle-point method and to demonstrate that our solution does not violate probability conservation.

Let us consider two extreme cases of the initial density distribution. First suppose that the initial density $\rho_0(\nu)$ is a smooth function, with a gradient of order unity, whence $S_0(\nu)$ vanishes. We would have $p_0 = 0, a = 1$ which means that the characteristic lines are the overdamped-pendulum trajectories (2.13). The time evolution (2.26) then becomes the fully classical one,

$$\rho(\mu, \tau) = \frac{1 - \nu^2}{1 - \mu^2} \rho(\nu, 0) \Big|_{\nu=\nu_{class}(\mu, \tau)}. \quad (2.29)$$

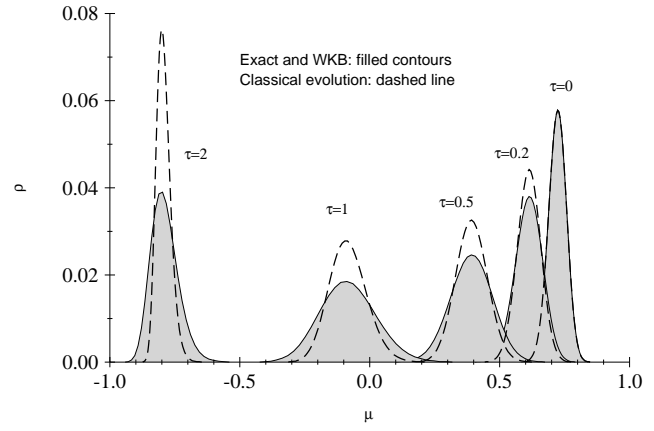


Fig. 1. Snapshots of the probability distribution $\rho(\mu, \tau)$ at various times, for the initially pure coherent state ($\gamma = 0.4, j = 200$). The WKB (Eq. (2.26)) and exact results shown by filled contours coincide in the scale of the plot. Dashed contours correspond to the classical evolution formula (2.29) based on the dynamics of the overdamped top. It grossly underestimates the width and overestimates the height of the peaks.

The speed of probability transport in this case is obtained by putting $a = 1$ in (2.12); since that speed depends on the coordinate the initial density distribution will change its form in time, due to its finite spatial size. In particular, that change involves broadening or sharpening depending on whether the distribution resides mostly over positive or negative values of μ , respectively.

Now consider the opposite extreme of a narrow initial density, perhaps one almost resembling a delta function. Then different parts of the packet will have practically the same coordinate but highly different momenta since $p_0 \approx \frac{1}{J\rho_0} \frac{\partial \rho_0}{\partial \nu}$. The maximum has $p_0 = 0$ and thus moves along an overdamped-pendulum trajectory. However the parts on the left and right slope have, respectively, $p_0 > 0$ and $p_0 < 0$. They will be transported along the Hamiltonian trajectories with $a > 1$ and $a < 1$, respectively. But according to (2.12) that means that the density on the left slope travels in the direction of negative μ faster than the one on the right slope. This will result in a spreading of the initially narrow distribution. There is an obvious superficial analogy with the decay of a wavepacket described by the Schrödinger equation. Of course, in our dissipative case the exponential factor does not describe any dephasing but rather a suppression of probability propagation along trajectories too strongly different from the fully classical overdamped-pendulum ones. This puts a brake on the spreading as soon as the packet widens; quantitative estimates of the width will be given below in the discussion of the properties of the propagator.

As an example, in Figure 1 we demonstrate the time dependence of the probability distribution in the case when the system was initially in a pure coherent state of the angular momentum $|\gamma\rangle$ (for properties of such states see *e.g.* [6, 7]). The parameter γ determines the direction of

the mean spin vector $\langle \gamma | \mathbf{J} | \gamma \rangle$ as $\gamma = \tan \frac{\theta_0}{2} e^{i\phi_0}$. We took $j = 200, \gamma = 0.4$. Three types of results are presented using:

1. numerical integration of the master equation (“exact values”);
2. the WKB solution (2.26);
3. the fully classical evolution formula (2.29).

The probability distributions given by the WKB formula coincide with the exact ones with accuracy in the range 0.4%–1.7% (accuracy decreases at the later stages of the evolution). The corresponding plots are indistinguishable. On the other hand, the fully classical formula correctly places the probability peaks but grossly underestimates their broadening with time leading to 20%–150% error in the width and amplitude; this error does not diminish as j grows(!)

2.7 The semiclassical dissipative propagator

The dissipative propagator establishes a linear relation between the initial and final density matrix elements. In the limit of large J the sum in this relation can be replaced by an integral; using the classical variables μ, ν as arguments it can be written (case $k = 0$)

$$\rho(\mu, \tau) = \int_{-1}^1 d\nu D(\mu, \nu, \tau) \rho(\nu, 0) \quad (2.30)$$

with the function $D(\mu, \nu, \tau)$ related to the matrix $D_{mn}(\tau)$ as $D(\mu, \nu, \tau) = J D_{mn}(\tau) |_{m=J\mu, n=J\nu}$.

To obtain the propagator one has to solve the master equation with the δ -peak as the initial density distribution. Strictly speaking such an initial condition does not fall into the class (2.3) such that our solution of the Cauchy problem (2.26) is not directly applicable. It is easy, however, to extract the propagator out of (2.26) in a slightly roundabout way. The semiclassical solution of the dissipative problem in the form (2.3) points to an analogy between our master equation for the densities and a Schrödinger equation in imaginary time. In the spirit of that analogy we may consider the function $D(\mu, \nu, \tau)$ in (2.30) as the van Vleck propagator [8] which must have the structure

$$D(\mu, \nu, \tau) = R(\mu, \nu, \tau) e^{J\Phi(\mu, \nu, \tau)}, \quad (2.31)$$

with $\Phi(\mu, \nu, \tau)$ the action accumulated along the trajectory.

Our task is to establish the prefactor R . To do so let us substitute the initial density (2.3) and the semiclassical propagator in the form (2.31) into (2.30) and perform the integration in the saddle-point approximation. The maximum $\nu^* = \nu^*(\mu, \tau)$ of the exponent just defines the Hamiltonian trajectories in the form (2.21). The saddle-

point integration thus gives

$$\begin{aligned} \rho(\mu, \tau) &= R(\mu, \nu, \tau) \sqrt{\frac{2\pi}{J}} \\ &\times \left\{ -\frac{\partial^2 [\Phi(\mu, \nu, \tau) + S_0(\nu)]}{\partial \nu^2} \right\}^{-1/2} \\ &\times e^{J\Phi(\mu, \nu, \tau)} \rho(\nu, 0) \end{aligned} \quad (2.32)$$

where $\nu^*(\mu, \tau)$ should be substituted for ν . Comparing with (2.26) we find the prefactor,

$$R = \sqrt{\frac{J}{2\pi}} \left\{ -\frac{\partial^2 [\Phi(\mu, \nu, \tau) + S_0(\nu)]}{\partial \nu^2} \right\}^{1/2} \frac{\sqrt{\frac{a^2 - \nu^2}{a^2 - \mu^2}}}{\sqrt{\frac{\partial \mu(\tau, \nu)}{\partial \nu}}}. \quad (2.33)$$

A simpler form results once we realize from (2.21) the Jacobian to satisfy

$$\frac{\partial \mu}{\partial \nu} = -\frac{\frac{\partial^2}{\partial \nu^2} [\Phi(\mu, \nu, \tau) + S_0(\nu)]}{\frac{\partial^2 \Phi(\mu, \nu, \tau)}{\partial \nu \partial \mu}}. \quad (2.34)$$

For the propagator we thus find

$$D(\mu, \nu, \tau) = \left[\frac{J}{2\pi} \frac{\partial^2 \Phi(\mu, \nu, \tau)}{\partial \nu \partial \mu} \right]^{1/2} \sqrt{\frac{a^2 - \nu^2}{a^2 - \mu^2}} e^{J\Phi(\mu, \nu, \tau)}. \quad (2.35)$$

Compared with the van Vleck propagator for the Schrödinger equation [8] the preexponential in this expression contains an additional square root factor, the origin of which can be traced to the difference in the normalizing condition for wave functions and density matrices. It is system specific in as much as a is a solution of (2.10). Note, however, that on the classical trajectory ($a = 1$) this factor is just the square root of the classical jacobian $d\mu^{-1}(\mu)/d\mu$, with $\nu = \mu^{-1}(\mu)$ the inverted classical trajectory (2.13). One easily verifies that both square roots in (2.35) give rise to the same factor and combine to the jacobian to the power one, as it is necessary to guarantee probability conservation.

By expressing the mixed derivative of the action Φ in the preexponent in terms of the function $\Xi(\mu, \nu, \tau)$ introduced in (2.28),

$$\frac{\partial^2 \Phi(\mu, \nu, \tau)}{\partial \mu \partial \nu} = [(a^2 - \mu^2)(a^2 - \nu^2) \Xi(\mu, \nu, \tau)]^{-1} \quad (2.36)$$

we arrive at the final form of our semiclassical density propagator matrix D_{mn} ,

$$D_{mn}(\tau) = \frac{1}{(a^2 - \mu^2) \sqrt{2\pi J \Xi}} e^{J\Phi(\mu, \nu, \tau)} \Big|_{\mu=\frac{m}{J}, \nu=\frac{n}{J}}, \quad (2.37)$$

in which a must be read as the function $a = a(\mu, \nu, \tau)$ since a Hamiltonian trajectory is already uniquely determined by specifying the initial and final coordinates. While in general that function can be found only numerically from the Hamiltonian trajectories (2.9), certain limits do yield approximate analytical expressions (see below). Note that a is always larger than the larger of $|\mu|, |\nu|$.

The behaviour of the propagator solution is largely determined by the action Φ . Two situations deserve special mention. First suppose that the initial quantum number n is not close to j . Then we encounter a narrow packet centered around the “fully classical” final point $\mu_{pend}(\tau; \nu)$ where the maximum of Φ is located. Close to that maximum the propagator can be represented by the Gaussian

$$D_{mn}(\tau) \approx \frac{1}{d\sqrt{2\pi}} \exp \left[-\frac{(\mu - \mu_{pend})^2}{2d^2} \right]. \quad (2.38)$$

whose width $d(\nu, \tau)$ is determined by the second derivative of the action (2.27). If expressed in terms of the classical coordinate μ the width $d = [-J\Phi_{\mu\mu}(\mu_{pend}, \nu, \tau)]^{-1/2}$ goes to zero in the classical limit like $J^{-1/2}$, such that for many purposes the propagator (2.38) may even be identified with the δ -function.

A radically different situation is met with when the initial quantum number n is close or equal to j so that the initial coordinate ν tends to 1 when $J \rightarrow \infty$. The action accumulated along any Hamiltonian trajectory is then close to zero like $1 - \nu$ and so is its second derivative. Indeed, one easily checks what we already saw in I,

$$a \approx 1 - \left(1 - e^{-2\tau'}\right) \delta_\nu \quad (2.39)$$

with

$$\delta_\nu = 1 - \nu = (j - n)/J \quad (2.40)$$

where $\tau' = \tau - \tau_{class}(\mu, \nu)$ denotes a quantum time shift, *i.e.* the difference between the travel times along the Hamiltonian trajectories (2.9) and the overdamped-pendulum ones (2.13). The action then comes out as

$$\Phi \approx \delta_\nu \left(1 - 2\tau' - e^{-2\tau'}\right). \quad (2.41)$$

Obviously, the exponent $J\Phi$ tends to a finite function of the relative time τ' when J tends to infinity as $j - n$ is kept fixed. The exponential factor is then only slightly less than unity along a Hamiltonian trajectory. Thus the width of the distribution does not tend to zero but rather stays finite as $J \rightarrow \infty$.

2.8 Comparison of the semiclassical propagators

The semiclassical propagator obtained in I by the saddle-point method,

$$D_{mn}(\tau) = \frac{1}{(1 - \mu^2)\sqrt{2\pi J\Xi}} e^{J\Phi(\mu, \nu, \tau)} \quad (2.42)$$

intriguingly differs from our present (2.37) in two respects. First, instead of $a^2 - \mu^2$ in the prefactor, (2.42) contains $1 - \mu^2$. Second, in (2.42) and in the whole of I we connected the classical coordinate μ to the quantum number m by $\mu = \frac{m-1}{J}$, which differs from our present definition $\mu = m/J$ by the small shift $\Delta = J^{-1}$.

Let us show that these two changes in fact cancel each other to leading order in J^{-1} . Consider the form (2.37) of our present propagator and substitute $\mu = \mu' + \Delta$ with μ' the shifted argument $(m-1)/J$. If m is not very close to n the denominator in the prefactor is not small and one can neglect its change brought about by the replacement $\mu \rightarrow \mu'$. As regards the exponent we must exercise greater care: The large factor J obliges us to keep two terms in the expansion

$$\begin{aligned} J\Phi(\mu, \nu, \tau) &= J\Phi(\mu' + \Delta, \nu, \tau) \\ &= J\Phi(\mu', \nu, \tau) + \ln \frac{a^2 - \mu^2}{1 - \mu^2} + \mathcal{O}(\Delta). \end{aligned} \quad (2.43)$$

The logarithm arising here obviously modifies the prefactor just so as to bring about the propagator (2.42) of the previous paper. Therefore, for $J \rightarrow \infty$ and m not close to n , the two forms (2.42) and (2.37) of the propagator are equivalent.

If m approaches n the equivalence of the formulae (2.37) and (2.42) does not hold any more: the accuracy of (2.42) is higher. In the extreme cases when $n - m$ or $j - n$ or $j \pm m$ are of order unity or zero¹ the uniform approximation for the propagator should be used instead of either (2.37) or (2.42) (see Sect. 1).

3 Conclusion

The role of the WKB approximation in quantum mechanics as a bridge to classical mechanics is common knowledge. What is not fully appreciated, though, is its potential usefulness for dissipative problems described by master equations. The problem of spin damping in superradiance may serve as a good example.

All results of the present paper followed from the WKB ansatz (2.3). It led us to a Hamilton-Jacobi equation of a classical mechanical system with one degree of freedom. The two canonical variables satisfy two Hamilton equations which require two initial or boundary data (like the initial coordinate and the initial momentum or the initial and final coordinate) to define a trajectory. This contrasts with the trajectories of the overdamped pendulum model described by a single differential equation of first order which are uniquely defined by the initial coordinate only. The extended family of Hamiltonian trajectories suggests an intuitive explanation of the quantum broadening of an initially sharp distributions alien to the traditional classical model: Although all relevant classical trajectories have

¹ These cases are similar to the ground state of a quantum system and cannot be adequately described by the WKB method.

$$J \frac{\partial \rho(\eta; \mu, \tau)}{\partial \tau} = J^2 \sqrt{[1 - (\mu + \eta)^2 - (\mu + \eta)\Delta][1 - (\mu - \eta)^2 - (\mu - \eta)\Delta]} \rho(\eta; \mu + \Delta, \tau) - J^2 (1 - \mu^2 - \eta^2 + \mu\Delta) \rho(\eta; \mu, \tau) + \mathcal{O}(\Delta^2). \quad (\text{A.4})$$

the same initial coordinate, the initial momenta are different, and so are the final coordinate.

There is of course a basic difference between the spreading of wave functions and the dissipative propagation of probabilities. In the case of unitary evolution the probability amplitudes to arrive from a fixed initial point to different final points differ by the phase accumulated along the respective classical trajectory (that is, if we neglect the prefactor). Consequently, propagation along all classical trajectories starting from the same initial point takes place with a comparable probability. In our dissipative problem the exponent of the WKB solution is either zero or negative. This means that there is no equality among the Hamiltonian trajectories starting at the same initial point: the one which corresponds to vanishing action (the fully classical overdamped-pendulum trajectory) is privileged to contribute most. Probability propagation along trajectories significantly removed from that privileged one is effectively suppressed. As a result the final width of an initially sharp distribution remains small, unlike the eventually infinite dispersion of the wavepacket of a free particle. The exceptional character of the superradiant evolution starting from the highest energy levels is also easily understood. In that case the classical action is close to zero on all trajectories. Therefore the exponential factor does not limit any more the broadening of the distribution with time.

The necessity to extend the dynamics of the overdamped-pendulum model was recognized long ago, see the reviews [9, 10]. In [11] a family of trajectories described by two first order differential equations was considered (they were introduced as the characteristics in the propagation of quasiprobability distributions); the two variables there employed did not form a canonical pair, though; one of them was related to our present coordinate μ and the other to the transverse components of the spin.

Our present WKB approach has the merit of being easily extended to other problems in dissipative quantum mechanics. Apart from providing an intuitive qualitative picture involving a Hamiltonian equivalent it also provides a convenient analytic approximation for the dissipative propagator. In particular we used it to calculate the width of the final distribution in the case when the initial direction of the Bloch vector was not close to the z axis. This result is of little significance for the theory of superradiance itself. However, the master equation (1.3) becomes increasingly important as a model in investigations of quantum chaos in dissipative systems [12, 13]; disregard of the final width of the propagator solutions would lead to erroneous results there. We shall extend the present work to dynamics with chaos in a subsequent paper.

Appendix A: Propagation of coherences

No separate investigation of the coherence propagator $D_{mn}^k(\tau)$ is necessary because of the identity proven in I,

$$D_{mn}^k(\tau) = D_{mn}(\tau) \frac{\sqrt{Q_{m-k, n-k} Q_{m+k, n+k}}}{Q_{mn}} e^{k^2 \tau / J} \quad (\text{A.1})$$

with

$$Q_{mn} = \prod_{l=m+1}^n g_l = \frac{(j+n)!(j-m)!}{(j+m)!(j-n)!}. \quad (\text{A.2})$$

It is instructive, however, to consider the changes in our Hamilton-Jacobi formalism necessitated by nonzero k . The new quantum number k whose range goes to infinity when $j \rightarrow \infty$ must be accompanied with a macroscopic variable

$$\eta = \frac{k}{J}. \quad (\text{A.3})$$

The master equation (1.3) written with μ, ν as arguments now reads

See equation (A.4) above

A Hamilton-Jacobi ansatz

$$\rho(\eta; \mu, \tau) = A(\eta; \mu, \tau) e^{JS(\eta; \mu, \tau)} \quad (\text{A.5})$$

entails a chain of differential equations for the ‘‘action’’ S and the terms in the expansion of the amplitude A in powers of Δ . We shall examine only the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial \tau} + G(\mu) - F(\mu) \exp\left(\frac{\partial S}{\partial \mu}\right) = 0 \quad (\text{A.6})$$

where F and G denote the auxiliary functions

$$\begin{aligned} F(\mu) &= \sqrt{[1 - (\mu + \eta)^2][1 - (\mu - \eta)^2]}, \\ G(\mu) &= 1 - \mu^2 - \eta^2. \end{aligned} \quad (\text{A.7})$$

The previous Hamiltonian becomes extended to

$$H(\mu, p) = G(\mu) - F(\mu)e^p. \quad (\text{A.8})$$

Once more denoting the conserved value of H by E and introducing the constant a by the relation

$$a = \sqrt{1 - E - \eta^2} \quad (\text{A.9})$$

we obtain the canonical equation for the coordinate,

$$\dot{\mu} = -F e^p = -(a^2 - \mu^2), \quad (\text{A.10})$$

It coincides with (2.12), and its integration leads to exactly the same trajectories (2.9) as for the densities. The characteristic lines for the coherences propagation are the same as the ones for probability propagation.

There is one cardinal difference. For $k = 0$ we could single out the special trajectories with zero initial momenta. Only these were important in the case of smooth initial densities, according to the relation $p \approx \partial \ln \rho / J \partial \mu$. Since the initially vanishing momentum remained zero at $\tau \neq 0$, an initially smooth density remained smooth as long as τ was not too large (before the system reached its lowest energy level). Smooth density distributions therefore form a closed class, and the overdamped-pendulum trajectories are their routes of propagation.

In the case of coherences it is also possible to select the trajectories corresponding to smooth initial distributions or zero initial momenta: the parameter a should be chosen then according to

$$a^2 = 1 - \eta^2 - G(\nu) + F(\nu), \quad (\text{A.11})$$

but now these trajectories do not have the physical significance of the fully classical trajectories of the overdamped pendulum. The reason simply is that an initially vanishing momentum p will no longer be zero when $\tau \neq 0$. Indeed, unless $\eta = 0$ and $a = 1$, the momentum

$$\begin{aligned} p &= \ln \frac{G(\mu) - E}{F(\mu)} \\ &= \ln \frac{a^2 - \mu^2}{\sqrt{[1 - (\mu + \eta)^2][1 - (\mu - \eta)^2]}}, \end{aligned} \quad (\text{A.12})$$

with $\mu = \mu(\tau)$ can never be a constant. Remembering the momentum-density connection we conclude that an initially smooth distribution of coherences inevitably ceases to be smooth in the course of time. Thus there is no special class of characteristics responsible for transporting smooth distributions of coherences and leaving them smooth, hence no elementary relation like (2.29).

Our considerations throw light on the important question whether it is admissible to replace the master

equation by a first-order differential equation using approximations like

$$\rho_{m+1} \approx \rho_m + \frac{\partial \rho_m}{\partial m}. \quad (\text{A.13})$$

Such a replacement is justified if the respective elements of the density matrix are and remain smooth functions of m . It follows:

- In the case of density propagation the above replacement (A.13) is justified provided the initial density distribution is smooth.
- For coherences the replacement (A.13) is always illegal.

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